Hale School

Mathematics Specialist

Term 2 2019

Test 4 - Integration

SECTION ONE

Name: Solutions

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Instructions:

- **SECTION ONE: CAS calculators are NOT allowed**
- **External notes are not allowed**
- **Duration of SECTION ONE: 30 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

Question 1 **(9 marks)**

Determine the following integrals:

a) $\int 2x(3x+1)^2 \, dx$ (3 marks)

$= \int 2x(9x^2+6x+1)\,dx$

$= \int 18x^3 + 12x^2 + 2x \, dx$

$= \frac{18x^4}{4} + \frac{12x^3}{3} + \frac{2x^2}{2} + c$

$= \frac{9x^4}{2} + 4x^3 + x^2 + c$

Or $u = 3x+1$

$= \int \frac{2(u-1)}{3} \cdot u^2 \cdot \frac{du}{3}$

$= \frac{2}{9}\left(\frac{u^4}{4} - \frac{u^3}{3}\right) + c$

$= \frac{(3x+1)^4}{18} - \frac{(3x+1)^3}{27} + c$

✓ expand

✓ integrate

✓ $+c$

b) $\int \sin^2(2x)\, dx$ (3 marks)

$= \int \frac{1}{2}(1 - \cos 4x)\, dx$

$= \frac{1}{2}x - \frac{\sin 4x}{8} + c$

✓ identity

✓ '$4x$'

✓ integrations

c) $\int \frac{\sin 2x}{\cos^2 x}\, dx$ (3 marks)

$= \int \frac{\sin 2x}{\frac{1}{2}(1+\cos 2x)}\, dx$

$= 2\int \frac{\sin 2x}{1+\cos 2x}\, dx$

$= -2 \ln|1+\cos 2x| + c$

$= \int \frac{2\sin x \cos x}{\cos x \cos x}\, dx$

$= 2\int \frac{\sin x}{\cos x}\, dx$

$= -2\ln|\cos x| + c$

✓ trig identity

✓ $\ln |\;|$

✓ simplifies

Question 2 **(6 marks)**

Using the substitution $x = 2\cos u$ determine the following definite integral:

$$\int \frac{x}{\sqrt{4-x^2}}\,dx$$

$x = 2\cos u$

$dx = -2\sin u\,du$

✓ change integral constant

$$= \int \frac{2\cos u}{\cancel{2\sin u}} \times -\cancel{2\sin u}\,du$$

$\sqrt{4-x^2} = \sqrt{4-4\cos^2 u}$

$= \sqrt{4\sin^2 u}$

$= 2\sin u$

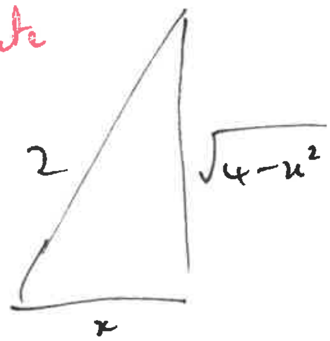
✓ simplify

$$= -\int 2\cos u\,du$$

✓ correct and simplify

$$= -2\sin u + c$$

✓ integrate



$$= -\frac{2\sqrt{4-x^2}}{2} + c$$

✓

replace u with x

$$= -\sqrt{4-x^2} + c$$

✓ correct final step

Question 3 **(8 marks)**

Determine the following integral:

$$\int \frac{2x^2-3}{x^2+x-6}\,dx = \int \frac{2(x^2+x-6)-2x+9}{(x^2+x-6)}\,dx$$

✓ splits numerator

$$= \int 2 + \frac{9-2x}{(x+3)(x-2)}\,dx$$

✓ factorises

But $\frac{9-2x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$

✓ sets up partial fractions

$\therefore \quad 9-2x = A(x-2) + B(x+3)$

✓ multiplies

$\therefore$ For $x=2$ $\quad 5 = 5B \quad B=1$

✓ Finds B

For $x=-3$ $\quad 15 = -5A \quad A=-3$

✓ Finds A

$$\therefore \int \frac{2x^2-3}{x^2+x-6}\,dx = \int 2 - \frac{3}{x+3} + \frac{1}{x-2}\,dx$$

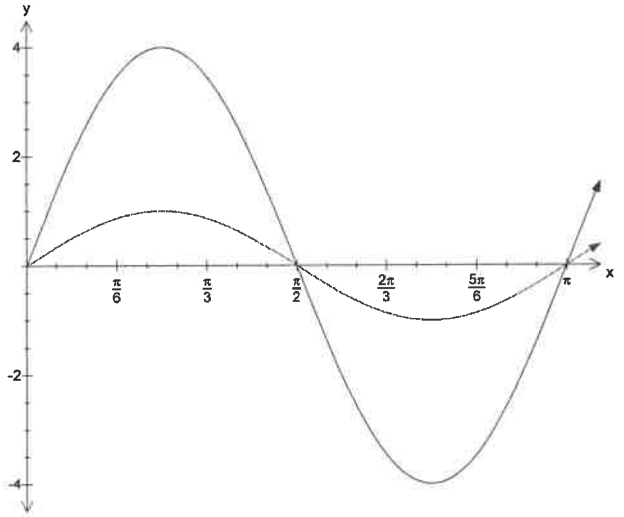
$$= 2x - 3\ln|x+3| + \ln|x-2| + c$$

✓✓ finds integral

$$= 2x + \ln\left|\frac{x-2}{(x+3)^3}\right| + c$$

Question 4 **(6 marks)**

The graph below shows two graphs, both of them of the form $y = a\sin bx$.



A solid is formed by rotating the area enclosed by the two graphs, one revolution about the X axis.

a) Show that the volume of the solid is given by

$$V_x = 15\pi\int_0^{\frac{\pi}{2}}(1-\cos 4x)dx$$ (4 marks)

$V_x = 2\times\pi\int_0^{\frac{\pi}{2}} y_2^2 - y_1^2\,dx$ ✓ $2\times\int_0^{\frac{\pi}{2}}$

$= 2\pi\int_0^{\frac{\pi}{2}}(4\sin 2x)^2 - (\sin 2x)^2\,dx$ ✓ function correct

$= 2\pi\int_0^{\frac{\pi}{2}} 16\sin^2 2x - \sin^2 2x\,dx$

$= 2\pi\int_0^{\frac{\pi}{2}} 15\sin^2 2x\,dx$

$= 2\pi\int_0^{\frac{\pi}{2}} 15\left(\frac{1-\cos 4x}{2}\right)dx$ ✓ identity

$= 15\pi\int_0^{\frac{\pi}{2}}(1-\cos 4x)\,dx$ ✓ expands to show result

b) Hence, use calculus methods to determine the **exact** volume of the solid. (2 marks)

$V = 15\pi\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$ ✓ integrates

$= \pi\left[\left(\frac{15}{2}\pi - 0\right) - (0-0)\right]$ ✓ sub

$= \frac{15}{2}\pi^2$

Question 5 **(7 marks)**

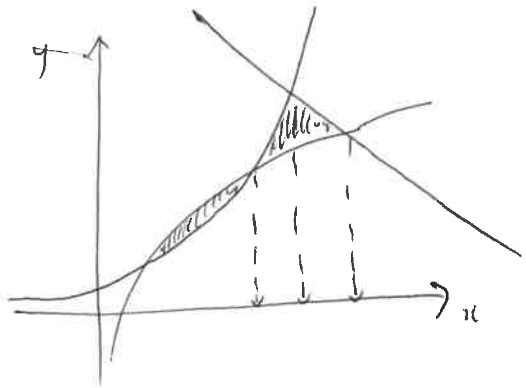
Encouraged by the success of the State of Origin, the two NSW second row forwards decide to move to WA to open a Fish and Chip shop. As keen mathematics students, they use calculus to devise a new two-dimensional logo for their shop.

The logo is to be solid blue and created by two areas.

The fish body is to be created by the area bound by the curves $f(x) = \ln(x) + 2$ and $f^{-1}(x) = e^{x-2}$.

The fish tail is to be created by the area enclosed by the curves $f(x) = \ln(x) + 2$, $f^{-1}(x)$ and the line $y = 10 - x$.

a) Determine an integral for the area of the fish body and evaluate. (4 marks)



$$\text{Area} = \int_{0.1586}^{3.1462} (\ln x + 2) - e^{x-2}\, dx$$

$$= 3.898 \text{ unit}^2$$

$\left.\begin{array}{l} y = \ln x + 2 \\ y = e^{x-2} \end{array}\right\}$ meet at $x = 0.1586$, $x = 3.1462$

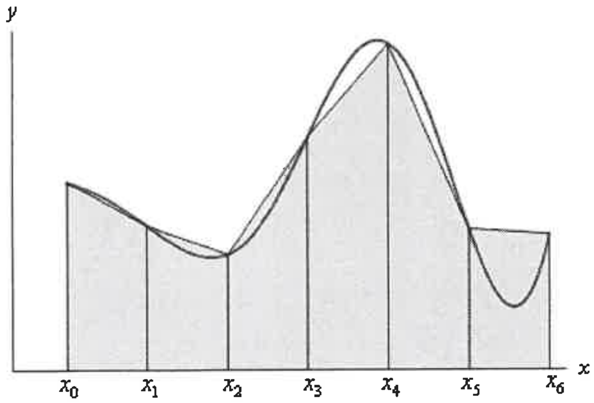
✓ upper bound
✓ lower bound
✓ integral
✓ answer

b) Establish an integral for the area of the fish tail (do not evaluate). (3 marks)

$$\int_{3.1462}^{3.8211} e^{x-2} - (\ln x + 2)\, dx + \int_{3.8211}^{6.1768} (10 - x) - (\ln x + 2)\, dx$$

✓ 1st integral
✓ 2nd integral
✓ bounds

Question 6 **(7 marks)**



The shaded region in the diagram above shows the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = x_0$ and $x = x_6$.

a) If $y_n = f(x_n)$, use the diagram above to show that the area using the trapezium rule with 6 strips is given by $A = \frac{1}{2}(x_1 - x_0)(y_0 + y_6 + 2\sum_{i=1}^{5} y_i)$. (3 marks)

width $= x_1 - x_0$

$$A = (x_1 - x_0) \times \left(\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \frac{y_2 + y_3}{2} + \frac{y_3 + y_4}{2} + \frac{y_4 + y_5}{2} + \frac{y_5 + y_6}{2} \right)$$

$$A = (x_1 - x_0) \times \left(\frac{y_0}{2} + \frac{y_6}{2} + y_1 + y_2 + y_3 + y_4 + y_5 \right)$$

$$A = \frac{x_1 - x_0}{2} \times \left(y_0 + y_6 + 2 \sum_{i=1} y_i \right)$$

✓ width
✓ $\frac{y_0 + y_1}{2}$ etc
✓ simplifies

b) For the area under the curve $y = 2x^3 - 6x^2 + 11x + 3$ from $x = 2$ to $x = 6$

(i) Find an approximation using the trapezium rule for 12 strips (1 mark)

$A = 413.33$ ✓

(ii) Find an approximation using Simpson's rule with 12 strips (1 mark)

$A = 412$ ✓

(iii) Comment on the accuracy of each approximation compared to the exact integral. (2 marks)

Simpson's rule gives an exact answer as it is a cubic ~~but~~ the trapezium rule is 0.32% above the true value. ✓✓

END OF TEST